THE DOPPLER EFFECT

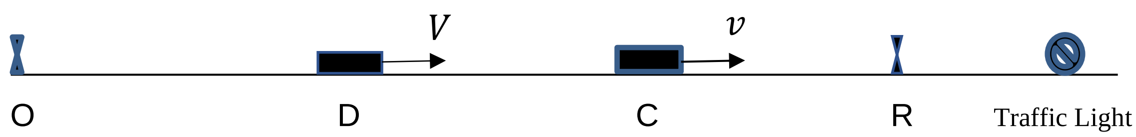
Preamble

The usual derivations of the relativistic Doppler formula follow from a prior development of the Lorentz transformation and the basic equations of special relativity, often with reference to exotic spaceships, rockets and their like. However, an alternative and delightfully intuitive approach to special relativity, the 'k-calculus' formulated by H. Bondi in 1962, begins with the relativistic Doppler effect illustrated with the aid of space-time diagrams, from which the other equations of special relativity are subsequently derived. In this note, we also start with Doppler effect, but rather than appealing to abstract diagrams or imaginary spaceships travelling at great speed, we refer here only to LiDAR and Radar traffic speed-traps which will be familiar to most people from first-hand experience. The discussion is based on the Principle of Relativity which states that the laws of physics are the same in all inertial frames, including the fact that the velocity of light is absolute. In other words, the mathematical formulation of the laws of physics will always have the same expression in terms of the coordinates of the inertial frame under consideration. In the final section, the formula for the classical (Newtonian) Doppler effect for sound waves is discussed and shown to be derived from a general relativistic formula that represents the Doppler effect for all cases.

LiDAR

Although Lidar is technologically sophisticated, it is based on the simple principle that a sequence of laser pulses emitted at given time intervals by a hand-held source will be reflected by a vehicle moving relative to the source with slightly different time intervals because the vehicle retreats from or advances towards the next pulse during each interval. The small difference between the intervals of the emitted and received signals enables the vehicle speed to be calculated.

In practice, the Lidar gun emits about two hundred pulses per second, but it is only necessary to consider two successive signals in order to analyse the principle on which it operates. The diagram here illustrates the situation we shall consider.



Suppose that a police officer O, stationed at position $x = 0$, observes a car speeding past him at time $t = 0$. For convenience we assume that the time t' measured by C, the driver of the car, as he passes O is also $t' = 0$. After the car has passed, the officer points the Lidar gun at the retreating vehicle. We assume that it emits a laser pulse at $t = t_1$ which is reflected from the retreating vehicle and received back by O at $t = \hat{t}_1$ and that the second pulse follows the first at time t_2 with its reflection received at time $\hat{t}_2$. Thus the time interval, or period, between the emitted pulses is $T = t_2 - t_1$, and that between the received pulses is $\hat{T} = \hat{t}_2 - \hat{t}_1$. Obviously, $\hat{T} > T$ because the car is moving away from O so that the second pulse has a slightly longer distance to travel than the first.

Since the speed of light c is the same in both directions, O will deduce that the distance d_1 of the car when the first pulse is reflected from it, is half the total distance travelled by the pulse there and back, i.e. $d_1 = ½c(\hat{t}_1 - t_1)$. The time τ_1 of the reflection, according to O, is the time when the pulse was emitted plus half its total travel time, i.e. $\tau_1 = t_1 + ½(\hat{t}_1 - t_1) = ½(t_1 + \hat{t}_1)$. Likewise, the second pulse is reflected from the car at time $\tau_2 = ½(t_2 + \hat{t}_2)$ when it is at a distance $d_2 = ½c(\hat{t}_2 - t_2)$ from O. In the time interval

$$\tau_2 - \tau_1 = ½(t_2 + \hat{t}_2 - t_1 - \hat{t}_1) = ½(\hat{T} + T)$$

between successive reflections, the car travels a distance

$$d_2 - d_1 = ½c(\hat{t}_2 - t_2 - \hat{t}_1 + t_1) = ½c(\hat{T} - T).$$

Its velocity v moving away from O is therefore given by

$$v = \frac{d_2 - d_1}{\tau_2 - \tau_1} = \frac{c(\hat{T} - T)}{\hat{T} + T}$$

which can be rearranged as

$$\frac{\hat{T}}{T} = \frac{1 + v/c}{1 - v/c}$$

Since $v \ll c$, this formula reduces to $\hat{T}/T \approx 1 + 2v/c$ to 1st order in v/c. Thus a simpler but very accurate expression for the speed of the receding car is

$$v \approx ½c\left(\hat{T}/T - 1\right).$$

Radar

Radar speed guns operate on a different principle. Instead of laser pulses it emits a continuous radio wave of frequency f, typically measured in bands either around 24 GHz or 34 GHz. After reflection the frequency of the received wave has undergone a frequency change. It is from this so-called Doppler frequency shift that the speed of the car is calculated.

Fortunately, the derivation in the previous section can be carried over intact with only a minor difference in interpretation. Thus we now regard T as the period of the emitted radio wave of frequency $f = 1/T$. Clearly $f > \hat{f} = 1/\hat{T}$ so that the Doppler frequency shift is $\Delta f = f - \hat{f}$ given by

$$\Delta f/f = T(1/T - 1/\hat{T}) = 1 - T/\hat{T} = 1 - (1 - v/c)/(1 + v/c).$$

It follows that

$$\frac{v}{c} = \frac{\Delta f/f}{2 - \Delta f/f}$$

so that $v \approx c\Delta f/2f$ to 1st order in $\Delta f/f$.

Modified results for an approaching vehicle

Imagine that a second police officer R is stationed far ahead of O on the long straight road along which the car is speeding (see diagram) and suppose that officer O asks officer R to use his Lidar gun to check the speed of the car from his vantage point as the car approaches him. Retaining the previous notation, we let T and $\hat{T}$ be the periods of the emitted and reflected pulses measured by R respectively. We note that $\hat{T} < T$ when the car is approaching, because successive pulses now have shorter paths to travel. By the same arguments as before, the time interval between the reflections of the two pulses from the car is still ½$(\hat{T} + T)$ but since the car is approaching, the distance it travels in this time is now ½$c(T - \hat{T})$. Thus corresponding to the formulae for a receding vehicle, we have

$$\frac{v}{c} = \frac{T - \hat{T}}{T + \hat{T}} \quad \text{and} \quad \frac{\hat{T}}{T} = \frac{1 - v/c}{1 + v/c}$$

Not surprisingly, these formulae are the same as the previous results with v replaced by $-v$.

Bondi's k-calculus

The preceding results have all been derived for the inertial frame of reference in which O and R are stationary. Henceforth we shall call this the rest frame. Consider now the timing of these events as measured by the driver C of the car. We shall label all times and positions observed by C with primed coordinates. In his frame of reference, he will receive the pulses at intervals of P', which for a receding vehicle must be greater than T, so that $P' = kT$ where $k > 1$. (Note that P, without the prime, would denote the time interval between pulses received by C as observed by O, not C.) Now from C's perspective, the signal is reflected with period P' back towards O who appears to be receding with speed v, thereby exactly mirroring the situation in the rest frame of reference because the speed of light is the same in all inertial frames. We may assert, therefore, that in his rest frame, O receives the reflected signal at intervals of kP'. This implies that

$$\hat{T} = kP' = k(kT) = k^2 T$$

Substituting the earlier result for a receding car we obtain

$$k = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

This is Bondi's k-factor. It expresses the Doppler effect for a receding source because the period $\hat{T}$ between the reflected pulses received by stationary observer O is a factor k greater than the period P' of their source from the moving car. Of course, it is equally true that the k-factor gives the Doppler effect for a receiver moving away from the source because P' is a factor k greater than T. Only their relative velocity of separation v is relevant.

Now consider the Doppler effect measured by officer R for whom the source is approaching. Let the corresponding factor be q, i.e. pulses originating from C with period P' are received by R with period qP'. This situation would arise if the pulses from O are partially transmitted from the car C to officer R, as well as being reflected back to O. But we know that R must receive them with period T because the two police officers O and R are stationary in the same rest frame. It follows that $T = qP' = qkT$, whence $q = 1/k$. This result obviously holds also for a receiver moving towards the source.

The Lorentz transformation

Suppose now that when officer O spots C speeding by, he decides to activate a warning traffic light at some distance x ahead, by sending a separate laser pulse which turns the traffic light on at time t. This means he must send the pulse at time $t - x/c$ because the activation pulse takes time x/c to reach the traffic light. The interval between the initial time 0 and the time of sending the pulse is therefore $T_0 = t - x/c$ which may be regarded as a time interval just like the interval between successive Lidar pulses discussed previously. The initial time 0 in C's reference frame is coincident with time 0 in O's frame because both O and C set their time to zero at the instant C passed O. According to the k-calculus, therefore, the time interval P_0' observed by C at the instant the activation signal passes the moving car is $P_0' = kT_0 = k(t - x/c)$. If, according to C, the traffic light is switched on at time t' when it is at a distance x' from the car, then by the argument we used for the rest frame of O, we deduce that $P_0' = t' - x'/c$. Hence

$$t' - x'/c = k(t - x/c)$$

When the traffic light is switched on, its light wave reaches C at time $t' + x'/c$ and continues onwards to reach O at time $t + x/c$. These two times can also be recognised as time intervals from the initial time 0 common to both frames of reference, so that the k-calculus gives

$$t + x/c = k(t' + x'/c).$$

Multiplying the first of these two equations by k and then adding and subtracting them, we obtain

$$2kct' = (k^2 + 1)ct + (k^2 - 1)x, \quad 2kx' = (k^2 + 1)x - (k^2 - 1)ct.$$

It follows by substitution for k and rearrangement, that

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

which is the well-known Lorentz transformation linking the coordinates (x, t) of an event in an inertial frame to the corresponding coordinates (x', t') of the same event in another inertial frame moving with relative velocity v.

In a system of units in which $c = 1$, i.e. all velocities are measured as fractions of c and distances are expressed in units of time (the time it takes for light to travel that distance), the Lorentz transformation takes on the symmetric form

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}, \quad t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

Velocity transformation

Suppose officer O summons a police car D to give chase to the speeding vehicle C with velocity V (see preceding diagram). If D is cruising at this speed between times t_3 and t_4 when it has reached distances x_3 and x_4 from O respectively, then

$$V = \frac{x_4 - x_3}{t_4 - t_3}$$

The corresponding coordinates in C's frame of reference are (x_3', t_3') and (x_4', t_4'). Substituting from the Lorentz transformation and dividing numerator and denominator by $t_4 - t_3$, we obtain an expression for V', the velocity of the police car according to C, as follows

$$V' = \frac{x_4' - x_3'}{t_4' - t_3'} = \frac{x_4 - vt_4 - x_3 + vt_3}{t_4 - vx_4/c^2 - t_3 + vx_3/c^2} = \frac{V - v}{1 - vV/c^2}$$

If D is parked at rest in O's frame of reference, then $V = 0$ and $V' = -v$, i.e. from C's perspective, O and D are receding backwards with velocity v, as expected. If $V = c$, then $V' = c$ confirming the invariance of the speed of light in all inertial frames. Note also that when $V \ll c$ and $v \ll c$, we recover the classical result $V' = V - v$ for relative velocities.

The Doppler effect for sound waves

Suppose also that the police car D switches on its siren while chasing the speeding car C. The signal from the siren propagates as a sound wave which differs from the radar or Lidar signals that travel at the speed of light in all inertial frames. Sound requires a medium in which to propagate. Its measured speed s is relative to the host medium, in this case the air. If we assume there is no wind blowing, the air is stationary in the rest frame and s is the speed of the sound wave as measured by observers on the ground.

The coordinates of events in other inertial frames of reference are still governed by the Lorentz transformation, of course, because communication between observers in different inertial frames can only made at the speed of light. Although such relativistic corrections are negligible when sound waves are being discussed, we shall retain the full equations for the time being. Appropriate approximations will be deferred until the end.

Let the times t_3 and t_4 in the last section be chosen so that they span exactly one period of the sound wave emitted by the siren, i.e. $t_4 - t_3 = T_a$ where the subscript a indicates that this period applies to an acoustic wave. Since car D is travelling with velocity $V < s$ it follows that $x_4 - x_3 = VT_a$. Note that all these quantities are defined in the rest frame. We denote their values in the inertial frame fixed on car D by an asterisk. Thus according to the Lorentz transformation, the actual period of the sound wave defined in the inertial frame of its source, is

$$T_a^* = t_4^* - t_3^* = \frac{t_4 - t_3 - V(x_4 - x_3)/c^2}{\sqrt{1 - V^2/c^2}} = \frac{T_a - V^2 T_a/c^2}{\sqrt{1 - V^2/c^2}} = T_a\sqrt{1 - V^2/c^2}$$

This is an example of the familiar time-dilation effect whereby the time interval T_a^* between two events appears longer (here T_a) to an observer in a frame that is moving relative to the inertial frame in which the events take place. The sound wave propagates with speed $s > v$ in the rest frame towards car C. Keeping the earlier notation, we denote by τ_3 the time that sound leaving the siren at time t_3 reaches C. At this time, C has travelled a distance $d_3 = v\tau_3$ so that the sound emitted at time t_3 takes a further time $(v\tau_3 - x_3)/s$ to travel to the car. It follows that $\tau_3 = t_3 + (v\tau_3 - x_3)/s$, whence $(s - v)\tau_3 = st_3 - x_3$. By a similar argument we deduce that $(s - v)\tau_4 = st_4 - x_4$. Subtraction gives

$$P_a = \tau_4 - \tau_3 = \frac{s(t_4 - t_3) - (x_4 - x_3)}{s - v} = \frac{s - V}{s - v} T_a = \frac{s - V}{s - v} \cdot \frac{T_a^*}{\sqrt{1 - V^2/c^2}}$$

where P_a is the period (in the rest frame) of the sound wave arriving at car C. In the same way the formula connecting T_a^* with T_a was derived earlier, time dilation gives $P_a' = P_a\sqrt{1 - v^2/c^2}$, which substituted in the equation above, yields

$$\frac{P_a'}{T_a^*} = \frac{s - V}{s - v}\sqrt{\frac{1 - v^2/c^2}{1 - V^2/c^2}}$$

The Doppler effect is usually expressed in terms of frequencies rather than periods. We denote the source frequency of the sound wave by $f_s = 1/T_a^*$ and the frequency observed by the receiver as $f_r = 1/P_a'$. We also write $v_s = V$ and $v_r = v$ for the velocities of the source and receiver respectively and finally make the approximations $v \ll c$ and $V \ll c$, giving

$$\frac{f_r}{f_s} \approx \frac{s - v_r}{s - v_s}$$

which is the classical expression for the Doppler effect for the source moving towards the receiver, and the receiver receding from the source. The corresponding formulae for source or receiver moving in the opposite direction are obtained simply by changing the relevant sign of v_r or v_s.

Connection between the classical and relativistic Doppler effects

It is instructive to see what happens if we replace s by c, the speed of light, in the preceding equations. Dropping the subscripts a (since we are no longer discussing acoustic waves) and putting $s = c$ in the exact formulae above, we obtain after some routine algebra

$$\frac{P'}{T^*} = \frac{1 - V/c}{1 - v/c}\sqrt{\frac{1 - v^2/c^2}{1 - V^2/c^2}} = \sqrt{\frac{1 + \bar{v}/c}{1 - \bar{v}/c}} = \bar{k} \quad \text{where} \quad \bar{v} = \frac{v - V}{1 - vV/c^2}$$

Here $\bar{k}$ is the Doppler factor observed in the rest frame of O, relative to which both source and receiver are in motion. It depends on a single velocity $\bar{v}$ which is easily shown to take the same algebraic form in all inertial frames in accordance with the Principle of Relativity. Moreover, its definition is simply the velocity transformation formula that represents $\bar{v}$ as either the velocity of C from D's perspective or the negative (oppositely directed) velocity of D from C's perspective. It is therefore a measure of the relative velocity between receiver and source; indeed it reduces to the classical relative velocity $v - V$ when $v \ll c$ and $V \ll c$.

Some special cases are illustrative. **(i)** If $V = 0$, then $\bar{v} = v$, $\bar{k} = k$ and $T^* = T$ giving $P' = kT$ which is the anticipated result because these are analogous conditions to those in our original example of a radar wave propagated from a stationary source O towards a receiver C receding with velocity v. **(ii)** If $V = v$, we have $\bar{v} = 0$, $\bar{k} = 1$ and $P' = T^*$ there being no Doppler effect when both C and D belong to the same inertial frame. **(iii)** Suppose $V < v$, so that car C is moving away from D. In D's inertial frame, C has a positive speed $v^* = \bar{v}$ by the velocity transformation formula. Thus $\bar{k} = k^*$ where k^* is the Doppler k-factor expressed in terms of the velocity v^*, and $P' = k^*T^*$ which is the Doppler effect in D's inertial frame that corresponds to the result in the rest frame of O given in example (i). **(iv)** If $V > v$, the source D is approaching receiver C. This is comparable to C approaching R in the rest frame as described earlier. The velocity of D expressed in C's inertial frame is $V' = -\bar{v}$ so that $\bar{k} = 1/k'$ and $P' = T^*/k'$ which is the expected Doppler effect for a source approaching the receiver.

Things are different with sound waves because the velocity of sound is not the same in all inertial frames. For example, although the air is stationary in the rest frame of O, the driver of car D will experience a headwind of velocity V. Thus the velocity of a soundwave in D's frame is $s^* = (s - V)/(1 - sV/c^2) \approx s - V$ since $V \ll c$ and $s \ll c$ and with the additional approximation $v \ll c$, the velocity of C is $v^* \approx v - V$ while $P_a' \approx P_a$ and $T_a^* \approx T_a$. To the same approximation, the Doppler effect for the sound waves in the rest frame is $P_a/T_a = (s - V)/(s - v)$ as we have already seen. (For simplicity we are using the formulae for the classical Doppler effect which excludes the negligible relativistic factor, but the same conclusions are reached even if this factor is retained, albeit with a more complicated algebraic derivation.) Expressed in terms of s^* and v^* by the relations above, the Doppler effect in D's frame becomes $P_a/T_a = s^*/(s^* - v^*)$. This is indeed the correct result for a source at rest (as the car D must be in its own reference frame, of course) in accordance with the Principle of Relativity which requires the laws of physics to be consistent in all inertial frames. Because these formulae also involve the velocity of sound in the particular inertial frame under consideration, however, they do not depend solely on the relative velocity between source and receiver as in the case of electromagnetic waves.

The exact formula for the Doppler effect

A result of interest in the last section was that the analysis of the Doppler effect for sound waves yielded a formula that also became valid for radar (in fact for any electromagnetic wave) as the wave speed s approached the speed of light c. The relativistic factor in the exact formula, which was negligible for situations involving sound waves, ensured that the formula merged into the correct form for radar waves. This suggests that the exact formula we derived for sound waves can be re-stated in a form that will cover all applications.

Thus, in the conventional notation introduced in the last section, where f_s and f_r are the frequencies of the emitted and received waves respectively and v_s and v_r denote the speeds of the source and receiver relative to the medium of wave propagation in which the wave speed is s, the Doppler effect for all cases is expressed by the single formula

$$\boxed{\frac{f_r}{f_s} = \frac{s \pm v_r}{s \pm v_s}\sqrt{\frac{1 - v_s^2/c^2}{1 - v_r^2/c^2}}}$$

The choice of positive or negative signs has been included to accommodate all four of the possible combinations of directions of motion:

(i) When the source is approaching the receiver and the receiver is retreating from the source as in the example considered here (car D is following car C) both numerator and denominator in this formula have minus signs.

(ii) If the situation is reversed so that the receiver is approaching the source and the source is retreating from the receiver, then both numerator and denominator have plus signs.

(iii) For both source and receiver approaching each other, the numerator will have a plus sign and the denominator a minus sign.

(iv) For both source and receiver moving away from each other, the numerator will have a minus sign and the denominator a plus sign.

John Weaver, 2019